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PLANE STRESS PROBLEMS FOR COMPRESSIBLE MATERIALS

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Abstract—Within the context of unconstrained, finite elasticity, exact solutions will be obtained for a number of plane stress boundary-value problems. The inflation and azimuthal shearing of cylinders of the so-called Varga material will be considered. Additionally, a simple, static interpretation of the strong ellipticity condition will be given for these materials.

1. INTRODUCTION

Controllable deformations are deformations that can be sustained by an elastic material independently of the specific form of the material response (the strain energy function), i.e. controllable deformations can be supported by surface tractions alone. Ericksen (1955) has shown that homogeneous deformations are the only controllable deformations for compressible, isotropic, elastic materials. However, this result does not preclude the possibility of non-homogeneous controllable deformations for specific compressible materials. Noteworthy in this regard are the solutions for harmonic materials (John, 1960) obtained by Ogden (1984), Ogden and Isherwood (1978), Abeyaratne and Horgan (1984) and Jafari *et al.* (1984).

Controllable deformations for other compressible materials have been investigated. Materials of type III were introduced by Carroll (1988) and independently by Haughton (1987). Materials of type III have a strain energy function of the form

$$W = c_1(i_1-3)+c_2(i_2-3)+h(i_3), \tag{1}$$

where the i_k are the principal invariants of the stretch tensor, the c_k are constants and h is an arbitrary function. Carroll (1988) has shown that both spherical expansion or compaction and cylindrical expansion or compaction, accompanied by an axial stretch, are controllable deformations for materials of type III. In a recent paper, Carroll and Murphy (1994) have shown that azimuthal shearing deformation is controllable for materials of type III, amongst others.

Boundary-value problems associated with such controllable deformations have also been studied in the literature. For cylindrical bodies of compressible materials, plane strain conditions are often assumed [see, for example, Jafari *et al.* (1984) and Carroll and Murphy (1994)]. Within this context, exact solutions to boundary-value problems can be obtained. The corresponding plane stress boundary-value problems cannot usually be solved exactly and, in this situation, the plane stress conditions are typically satisfied in a resultant sense.

In this paper, we will solve a number of plane stress boundary-value problems in an exact manner for a restricted (yet still quite general) material of type III. This material has a form of the strain energy function equal to (1) above with c_2 set equal to zero. This is the Varga material introduced by Haughton (1987). Azimuthal shearing and cylindrical

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inflation of this material yield constant out-of-plane stress components. Consequently the plane stress problem can be solved exactly.

There is no obvious reason why the strain energy function of a real material should be of the special form corresponding to materials of type III. A number of constitutive restrictions on the form of the strain energy function for materials of type III will first be considered after the preliminaries. A number of plane stress boundary-value problems will then be examined. These problems will involve cylindrical inflation and azimuthal shearing.

2. SPECIAL STRAIN ENERGY FUNCTIONS

The response of an elastic material is completely described by the form of its strain energy function:

$$W = \hat{W}(\mathbf{F}), \tag{2}$$

where $\mathbf{F}$ is the deformation gradient tensor satisfying

$$\det \mathbf{F} > 0. \tag{3}$$

$\mathbf{F}$ has the polar decompositions

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{4}$$

where the rotation $\mathbf{R}$ is a proper orthogonal tensor and the stretch tensors $\mathbf{U}$, $\mathbf{V}$ are positive definite and symmetric.

Invariance under superposed rigid body motions leads to:

$$W = \hat{W}(\mathbf{U}). \tag{5}$$

The assumption of material isotropy further leads to

$$W = \bar{W}(i_1, i_2, i_3), \tag{6}$$

where i_1, i_2, and i_3 are the usual invariants of the stretch tensor $\mathbf{U}$ (and of $\mathbf{V}$, since $\mathbf{U}$, $\mathbf{V}$ have identical invariants).

The stress response equations

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}, \quad \mathbf{T} = i_3^{-1}\mathbf{P}\mathbf{F}^T, \tag{7}$$

where $\mathbf{P}$, $\mathbf{T}$ are the Piola and Cauchy stress tensors, then lead to a representation

$$\mathbf{T} = \frac{\partial W}{\partial i_3}\mathbf{1} + i_3^{-1}\left(\frac{\partial W}{\partial i_1} + i_1\frac{\partial W}{\partial i_2}\right)\mathbf{V} - i_3^{-1}\frac{\partial W}{\partial i_2}\mathbf{V}^2. \tag{8}$$

The Cayley–Hamilton theorem

$$\mathbf{V}^3 - i_1\mathbf{V}^2 + i_2\mathbf{V} - i_3\mathbf{1} = 0 \tag{9}$$

may be used to express $\mathbf{V}^2$ as a linear combination of $\mathbf{1}$, $\mathbf{V}$, and $\mathbf{V}^{-1}$. Substitution for $\mathbf{V}^2$ in (8) leads to the alternative representation

$$\mathbf{T} = i_3^{-1}\left(i_2\frac{\partial W}{\partial i_2}+i_3\frac{\partial W}{\partial i_3}\right)\mathbf{1}+i_3^{-1}\frac{\partial W}{\partial i_1}\mathbf{V}-\frac{\partial W}{\partial i_2}\mathbf{V}^{-1}. \tag{10}$$

Carroll (1988) introduced the so-called materials of type III whose strain energy function has the form

$$W = c_1(i_1-3)+c_2(i_2-3)+h(i_3), \tag{11}$$

where c_1, c_2 are constants and h is a twice continuously differentiable function. Substitution of (11) into the stress–strain relation (10) gives

$$\mathbf{T} = c_1 i_3^{-1}\mathbf{V}+c_2(\mathrm{tr}\,\mathbf{V}^{-1}\mathbf{1}-\mathbf{V}^{-1})+h'(i_3)\mathbf{1}. \tag{12}$$

The conditions that the strain energy and the stress vanish in the reference configuration yield

$$h(1) = 0, \quad h'(1) = -(c_1+2c_2). \tag{13}$$

It is to be expected that a number of inequalities will have to be imposed on the arbitrary function $h(i_3)$ in order that the behavior of the solutions of the equilibrium equations conforms with physical intuition. Some such constitutive restrictions will be considered in the next section.

3. RESTRICTIONS ON THE STRAIN ENERGY FUNCTION

In the linear theory the elastic response is not arbitrary: the strain energy function must be positive definite in the (infinitesimal) strains to ensure physically realistic behavior. Since the large deformation theory includes the linear theory as a limiting case, one necessary restriction on the strain energy function is immediately obvious: on restriction to infinitesimal deformations, the shear and bulk moduli should be positive. Now, on restriction to infinitesimal deformations, the strain energy function of materials of type III reduces to the strain energy function of the linear theory if and only if

$$c_1+c_2 = 2\mu$$

$$h''(1) = \lambda+2\mu, \tag{14}$$

where λ and μ are the Lamé constants of the material satisfying the inequalities

$$3\lambda+2\mu > 0, \quad \mu > 0. \tag{15}$$

In the context of large deformations, one approach to ensure physically realistic response is through static considerations (Truesdell and Noll, 1965). An elastic material satisfies the Baker–Ericksen (BE) inequalities if and only if

$$(t_i-t_j)(\lambda_i-\lambda_j) > 0, \quad \text{if } \lambda_i \neq \lambda_j, \tag{16}$$

where t_i, λ_i are the principal Cauchy stresses and the principal stretches, respectively. An elastic material satisfies the strong tension–extension (STE) inequalities if and only if

$$\frac{\partial t_i}{\partial \lambda_i} > 0, \quad i = 1,2,3 \quad \text{(no sum)} \tag{17}$$

everywhere. The physical motivation behind these inequalities is to be found in Truesdell and Noll (1965).

We now consider the restrictions placed on the strain energy function of materials of type III by the above two inequalities. For materials of type III, the principal Cauchy stresses can be easily obtained from (11)

$$t_i = h'(i_3) + \frac{c_1}{\lambda_j \lambda_k} + c_2 \left(\frac{1}{\lambda_j} + \frac{1}{\lambda_k} \right), \quad i \neq j \neq k \neq i, \tag{18}$$

where $i_3 = \lambda_1 \lambda_2 \lambda_3$. Substitution of (18) into (16) shows that the BE inequalities are satisfied if and only if

$$c_1 + c_2 \lambda_k > 0, \quad k = 1, 2, 3. \tag{19}$$

The inequalities (19) are equivalent to the following condition:

$$\text{either } c_1 \geqslant 0, \quad c_2 > 0 \quad \text{or} \quad c_1 > 0, \quad c_2 \geqslant 0. \tag{20}$$

Substitution of (18) into (17) shows that for materials of type III, the STE inequalities are equivalent to

$$h''(i_3) > 0. \tag{21}$$

Another type of constitutive restriction that has received considerable attention in the literature is the strong ellipticity (SE) condition [see, for example, Ogden (1984), Truesdell and Noll (1965), Knowles and Sternberg (1975, 1977)]. In the formulation due to Ogden (1970), an elastic material satisfies the SE condition if and only if

$$Q_{ij}(\mathbf{n}) m_i m_j > 0 \tag{22}$$

for arbitrary, non-zero vectors **m**, **n** and where

$$Q_{11} = \lambda_1 \frac{\partial t_1}{\partial \lambda_1} n_1^2 + \frac{t_1 - t_2}{\lambda_1^2 - \lambda_2^2} \lambda_2^2 n_2^2 + \frac{t_1 - t_3}{\lambda_1^2 - \lambda_3^2} \lambda_3^2 n_3^2, \text{ etc.}$$

$$Q_{12} = \left(\lambda_2 \frac{\partial t_1}{\partial \lambda_2} + \frac{t_1 - t_2}{\lambda_1^2 - \lambda_2^2} \lambda_1^2 \right) n_1 n_2, \text{ etc.} \tag{23}$$

Substitution of (18) into (23), permutation of the indices in (23) and some algebra yields the following form of the SE condition for materials of type III

$$i_3^2 h''(i_3)(n_1 m_1 + n_2 m_2 + n_3 m_3)^2 + \frac{\lambda_1 c_2 + c_1}{\lambda_2 + \lambda_3} (\lambda_2 n_2 m_3 - \lambda_3 n_3 m_2)^2$$
$$+ \frac{\lambda_2 c_2 + c_1}{\lambda_1 + \lambda_3} (\lambda_1 n_1 m_3 - \lambda_3 n_3 m_1)^2 + \frac{\lambda_3 c_2 + c_1}{\lambda_1 + \lambda_2} (\lambda_1 n_1 m_2 - \lambda_2 n_2 m_1)^2 > 0. \tag{24}$$

From Truesdell and Noll (1965), we see that, for isotropic materials, both the BE and STE inequalities are necessary for strong ellipticity to hold. From (19), (21) and (24), we conclude that the BE and STE inequalities are also sufficient for the SE condition to be satisfied for materials of type III. This is a simple, static interpretation of the SE condition. We note that this interpretation is not valid in general. This can easily be seen from consideration of the special material introduced in Knowles and Sternberg (1975). In what follows, we will assume that the SE condition is satisfied.

4. AZIMUTHAL SHEAR

We consider the problem in which the inner radius of an annulus is held fixed while the outer surface is twisted through an angle. The hollow cylinder has original inner radius R_1, original outer radius R_2 and original length L. Using a semi-inverse approach, Carroll and Murphy (1994) found that the following controllable deformation describes such azimuthal shearing for materials of type III:

$$r^2 = \alpha R^2 + \beta, \quad \theta = \Theta + \sin^{-1}\left[\frac{A}{R\sqrt{\alpha R^2+\beta}}\right] + B, \quad z = \lambda Z, \tag{25}$$

where α, β, A, and B are constants and where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates of a particle before and after deformation, respectively.

We will now consider two associated boundary-value problems. The first will be the obvious displacement boundary-value problem where the outer surface is twisted through a given angle Ω at fixed radius. The second problem considered will be where a moment M is applied to the otherwise traction-free outer surface.

In both boundary-value problems we will require the planar ends of the cylinder to be stress free, i.e. we will consider plane stress problems. From (11) and (25) we see that

$$T_{rz} = T_{\theta z} = 0 \tag{26}$$

and that

$$T_{zz} = \frac{c_1}{\alpha} + c_2\left[\frac{2\alpha R^2+\beta}{\alpha\sqrt{R^2(\alpha R^2+\beta)-A}}\right] + h'(\lambda\alpha) \tag{27}$$

Setting $c_2 = 0$, the T_{zz} component of Cauchy stress becomes constant

$$T_{zz} = \frac{c_1}{\alpha} + h'(\lambda\alpha). \tag{28}$$

The interesting consequence of (28) is that we can satisfy the conditions of plane stress *exactly* for materials of type III with $c_2 = 0$ (the so-called Varga materials), rather than in the usual resultant (or St Venant) sense.

Thus, the deformation describing plane stress, azimuthal shearing of the Varga material is given by (25) where λ and α are related by

$$h'(\lambda\alpha) = -\frac{2\mu}{\alpha}, \tag{29}$$

where, from (14), $c_1 = 2\mu$. We now consider the associated boundary-value problems.

5. DISPLACEMENT BOUNDARY-VALUE PROBLEM

We consider the problem in which the inner radius of an annulus is held fixed while the outer surface is twisted clockwise through an angle Ω at fixed radius. The boundary conditions therefore are

$$r_1 = \hat{r}(R_1) = R_1, \quad \theta = \Theta \quad \text{on} \quad R = R_1 \tag{30}$$

$$r_2 = \hat{r}(R_2) = R_2, \quad \theta = \Theta - \Omega \quad \text{on} \quad R = R_2 \tag{31}$$

together with the plane stress condition (29).

The solution for α, β, A, and B proceeds exactly as in Carroll and Murphy (1994) and we obtain the following results

$$\alpha = 1, \quad \beta = 0$$
$$A = \frac{R_1^2 R_2^2 \sin\Omega}{(R_1^4 + R_2^4 - 2R_1^2 R_2^2 \cos\Omega^{1/2}},$$
$$B = -\sin^{-1}\left[\frac{A}{R_1^2}\right]. \tag{32}$$

Since $\alpha = 1$, (29) becomes

$$h'(\lambda) = -2\mu. \tag{33}$$

But from (13) and (14) we already have

$$h'(1) = -2\mu$$

and since $h''(\lambda) > 0$, by the STE inequalities, we immediately see that (33) has the unique solution

$$\lambda = 1.$$

Thus, for this special case we see that the plane strain and the plane stress problems are equivalent. In particular, the relationship between applied torque and angular displacement for the plane stress problem is the same as that given in Carroll and Murphy (1994) where the corresponding plane strain problem was considered. Thus, letting M denote the moment necessary to twist the outer cylindrical surface in a clockwise direction relative to the fixed inner surface, the following relationship holds

$$M_n = \frac{\gamma^2 \sin\Omega}{(1+\gamma^4 - 2\gamma^2\cos\Omega)^{1/2}}, \tag{34}$$

where M_n is the non-dimensionalized moment defined by

$$M_n = \frac{M}{4\pi\mu R_1^2 L} \tag{35}$$

and

$$\gamma^2 = \frac{R_2^2}{R_1^2}. \tag{36}$$

We note that (34) is valid for all materials of type III. Finally, letting $\gamma^2 \to \infty$ in (34), which corresponds to the azimuthal shearing of an infinite medium of type III with a rigid cylindrical insert, we obtain

$$M_n = \sin\Omega. \tag{37}$$

6. TRACTION BOUNDARY-VALUE PROBLEM

We next consider the problem in which the inner radius of an annulus is held fixed while a clockwise moment M is applied to the otherwise traction-free outer surface. The boundary conditions are

$$r_1 = \hat{r}(R_1) = R_1, \quad \theta = \Theta \quad \text{on} \quad R = R_1 \tag{38}$$

$$T_{rr} = 0, \quad \text{on} \quad R = R_2 \tag{39}$$

$$T_{r\theta} = -\frac{M}{2\pi r^2 \lambda L}, \quad R = R_2 \tag{40}$$

together with the plane stress condition (34).

We first note that (38) yields:

$$\beta = R_1^2(1-\alpha)$$
$$B = -\sin^{-1}\left[\frac{A}{R_1^2}\right]. \tag{41}$$

We now consider the stress boundary conditions. From (12) (with $c_2 = 0$) and (25), we see that the stress components T_{rr} and $T_{r\theta}$ are given by

$$T_{rr} = h'(\lambda\alpha) + 2\mu\frac{\sqrt{R^2(\alpha R^2+\beta)-A^2}}{\lambda(\alpha R^2+\beta)} \tag{42}$$

$$= 2\mu\left(-\frac{1}{\alpha}+\frac{\sqrt{R^2(\alpha R^2+\beta)-A^2}}{\lambda(\alpha R^2+\beta)}\right), \tag{43}$$

using (34) and

$$T_{r\theta} = -2\mu\frac{A}{\lambda r^2} \tag{44}$$

using $c_1 = 2\mu$ as before.

Application of (40) yields

$$\frac{A}{R_1^2} = \frac{M}{4\mu R_1^2 L\pi} = M_n \tag{45}$$

using the same notation as before. Applying (39) we obtain

$$\lambda = \frac{\alpha\sqrt{\gamma^2(\alpha(\gamma^2-1)+1)-M_n^2}}{\alpha(\gamma^2-1)+1}, \tag{46}$$

where

$$\gamma^2 = \frac{R_2^2}{R_1^2}$$

as before.

To determine α, we use the plane stress condition (29) and equation (46). We seek solutions $\alpha > 0$ of the equation

$$h'\left(\frac{\alpha^2\sqrt{\gamma^2(\alpha(\gamma^2-1)+1)-M_n^2}}{\alpha(\gamma^2-1)+1}\right) = -\frac{2\mu}{\alpha}. \tag{47}$$

Now let Ω denote the relative angular displacement of the two cylindrical surfaces. Then, using (41), we have

$$\Omega = \sin^{-1}\left[\frac{A}{R_1^2}\right] - \sin^{-1}\left[\frac{A}{R_2 r_2}\right], \tag{48}$$

where

$$r_2 = \sqrt{\alpha R_2^2 + \beta} = \sqrt{\alpha(R_2^2 - R_1^2) + R_1^2}.$$

Therefore, we get from (48)

$$\Omega = \sin^{-1}\left(M_n\left(1 - \frac{R_1^4}{R_2^2 r_2^2} M_n^2\right)^{1/2} - \frac{R_1^2}{R_2 r_2} M_n (1 - M_n^2)^{1/2}\right), \tag{49}$$

where

$$\frac{R_1^2}{r_2 R_2} = \frac{1}{\gamma\sqrt{\alpha(\gamma^2 - 1) + 1}} \tag{50}$$

and $\alpha = \hat{\alpha}(M_n)$ is determined from (47).

Therefore, we conclude from the above analysis that the solution of the given boundary-value problem is dependent on the form of the arbitrary function $h(i_3)$. In particular, the relationship between applied torque and angular displacement is dependent on the form of $h(i_3)$ through (47) and (50), in contrast to the displacement boundary-value problem.

However, for the special case of a rigid cylinder embedded in an infinite medium of the Varga material, we can obtain a relationship between applied torque and angular displacement which is independent of the form of the strain energy function. Letting $\gamma^2 \to \infty$ in (47), we obtain the following equation which determines α for this special case

$$h'(\alpha^{3/2}) = -\frac{2\mu}{\alpha}. \tag{51}$$

With α determined from (51) [for specified form of $h(i_3)$], λ is determined by

$$\lambda = \alpha^{1/2} \tag{52}$$

which equation is obtained on letting $\gamma^2 \to \infty$ in (46). On consideration of (50), we immediately see that

$$\lim_{\gamma^2 \to \infty} \frac{R_1^2}{r_2 R_2} = 0. \tag{53}$$

Consequently, as $\gamma^2 \to \infty$, we obtain, from (49), the familiar result

$$M_n = \sin\Omega. \tag{54}$$

An examination of (49) reveals that the range of applied moment M must be bounded to ensure physically realistic behavior. Specifically, we require as a necessary condition for physically realistic behavior that M_n satisfy

$$0 \leqslant M_n \leqslant 1. \tag{55}$$

A similar restriction on the range of the angle of twist in the displacement boundary-value problem can be obtained. Details can be found in Carroll and Murphy (1994).

7. CYLINDRICAL INFLATION

We again consider a hollow cylinder of type III material, with the same dimensions as before. Carroll (1988) has shown that the controllable deformation

$$r^2 = \frac{\alpha}{\lambda}R^2+\beta, \quad \theta = \Theta, \quad z = \lambda Z, \tag{56}$$

where α, β, and λ are constants and where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates of a particle before and after deformation, describes cylindrical inflation or compaction accompanied by an axial stretch for materials of type III.

We will consider the associated plane stress problem. Now from (12) and (56) we see that

$$T_{rz} = T_{\theta z} = 0 \tag{57}$$

and that

$$T_{zz} = c_1\frac{\lambda}{\alpha}+c_2\left(\frac{\lambda}{\alpha}\frac{r}{R}+\frac{R}{r}\right)+h'(\alpha). \tag{58}$$

Setting $c_2 = 0$, we obtain

$$T_{zz} = c_1\frac{\lambda}{\alpha}+h'(\alpha). \tag{59}$$

This stress component is constant and we can therefore again satisfy the plane stress conditions exactly rather than in the resultant sense. We conclude from (59) that the plane stress conditions are satisfied exactly if and only if

$$h'(\alpha)+2\mu\frac{\lambda}{\alpha} = 0, \tag{60}$$

where $c_1 = 2\mu$ from (14).

Assume that both internal and external pressure are exerted on the curved surfaces of the cylinder. Then

$$\begin{aligned} T_{rr} &= -p_i, \quad \text{on } R = R_1 \\ T_{rr} &= -p_o, \quad \text{on } R = R_2. \end{aligned} \tag{61}$$

Thus for the Varga material, we get from (12) and (56) that

$$T_{rr} = 2\mu\frac{R}{\lambda r}+h'(\alpha) \tag{62}$$

and therefore equations (61) become

$$\begin{aligned} 2\mu\frac{R_1}{\lambda r_1}+h'(\alpha) &= -p_i \\ 2\mu\frac{R_2}{\lambda r_2}+h'(\alpha) &= -p_o, \end{aligned} \tag{63}$$

where $r_1 = \hat{r}(R_1)$, $r_2 = \hat{r}(R_2)$. Now from the first equation in (56) we obtain the identity

$$\gamma^2\left(\frac{r_2}{R_2}\right)^2-\left(\frac{r_1}{R_1}\right)^2=\frac{\alpha}{\lambda}(\gamma^2-1), \tag{64}$$

where

$$\gamma^2=\frac{R_2^2}{R_1^2}.$$

If we solve (63) for R_1/r_1 and R_2/r_2 and substitute into (64) we obtain

$$\frac{\gamma^2}{(\bar{p}_o+H(\alpha))^2}-\frac{1}{(\bar{p}_i+H(\alpha))^2}=\lambda\alpha(\gamma^2-1), \tag{65}$$

where

$$\bar{p}_o=\frac{p_o}{2\mu},\quad \bar{p}_i=\frac{p_i}{2\mu},\quad H(\alpha)=\frac{h'(\alpha)}{2\mu}. \tag{66}$$

Now if we eliminate λ between (60) and (65) we obtain

$$\frac{1}{(\bar{p}_i+H(\alpha))^2}-\frac{\gamma^2}{(\bar{p}_o+H(\alpha))^2}=\alpha^2H(\alpha)(\gamma^2-1). \tag{67}$$

We require specification of the form of the arbitrary function $h(i_3)$ in order to obtain a solution to the given boundary-value problem. Assuming that $h(i_3)$ has been specified, we obtain $\alpha>0$ from (67), $\lambda>0$ from (60) and β from (63).

Some simplification of the above equations occurs for the case of an infinite sheet with a circular hole. This corresponds to the limiting case of R_2 tending to infinity. Taking the required limits in (67), we obtain the following equation to determine α

$$\alpha^2H(\alpha)+\frac{1}{(\bar{p}_o+H(\alpha))^2}=0, \tag{68}$$

where $\bar{p}_o$ is now the applied pressure at infinity. The second of (63) now becomes

$$\frac{1}{(\lambda\alpha)^{1/2}}+H(\alpha)=-\bar{p}_o. \tag{69}$$

Substitution for $H(\alpha)$ from (60) yields the equation to determine λ, once α has been determined from (68)

$$\frac{1}{(\lambda\alpha)^{1/2}}-\frac{\lambda}{\alpha}=-\bar{p}_o. \tag{70}$$

β is again determined from the first equation of (63).

We finally note that for internal pressure only, that is for $\bar{p}_o$ equal to 0, the above equations simplify further. Equation (68) becomes

$$\alpha^2H^3(\alpha)+1=0 \tag{71}$$

and (70) becomes

$$\lambda = \alpha^{1/3}. \tag{72}$$

We emphasize that this solution is exact although we require a specification of the arbitrary function $h(i_3)$ in order to obtain a solution to the given boundary-value problem. Finally, we notice that the solutions α, λ to (71) and (72) are independent of the pressure on the hole, $\bar{p}_i$. Since from the first equation of (63)

$$r_1 = -\frac{R_1}{\lambda}\frac{1}{H(\alpha)+\bar{p}_i} \tag{73}$$

we conclude that for physically realistic response, $\bar{p}_i$ must satisfy

$$0 \leqslant \bar{p}_i < -H(\alpha) \tag{74}$$

and that for $\bar{p}_i$ satisfying (74), we have

$$\frac{\mathrm{d}r_1}{\mathrm{d}\bar{p}_i} > 0. \tag{75}$$

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